Equichordal curves and their applications – the geometry of a pulsation-free pump

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Abstract

An almost forgotten class of closed plane curves, the class of equichordal curves, is re-introduced and a couple of interesting mathematical conjectures are suggested in this context. Since the requirements for being an equichordal curve are rather weak, there remains a remarkable amount of room to satisfy specific requirements and optimizations for technical applications requiring the features of this class of curves. This flexibility is the root of their technical applicability. As an example, a high pressure, high precision displacement pump with eliminated pulsation, is discussed in detail and some other possible applications are mentioned.

Key words: Plane curves; Equichordal curves; Convexity; Symmetries; Ordinary differential equation; Relaxation method; Rotary pump; Rotary internal combustion engine

1 Introduction

Of late, more and more functional mechanical parts appear in the field of machine design, which employ some unusual and somewhat complex geometrical shapes going beyond the geometry of traditional mechanics, the world governed by planar and cylindrically symmetrical surfaces. In this revolutionary process the equichordal curves – a barely known class of plane curves – could also play a significant role. This paper describes the most important features of this curve class. The author has found some realistic and delicate mechanical applications for these curves and their relatives, and also revealed a few mathematical conjectures, which – according to the knowledge of the author – are not proven so far and may claim for general interest.

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2 Equichordal curves

Since this fascinating class of closed plane curves is treated rather briefly and superficially in geometry textbooks, let us provide a summary of the basic features and a classification of these curves. One of the possible derivations of this curve class is a single step generalization of the circle, if the circle is defined as the locus of the endpoints of a section, while it rotates π radian (180 degrees) around its middle point that is also fixed to the plane. If we relax this definition only to the extent, that the center of rotation remains a fixed point of the plane, but it is not fixed relative to the section, thus the section may slip freely over the point of rotation while it turns around, and the only requirement is that after a rotation of π radian, the section should fully overlap its original position; the locus of the endpoints of the section is a general equichordal curve (see Fig. 1).

2.1 General definitions and features

Definition 1. *(Continuous equichordal curve)* a continuous equichordal plane curve is the locus of the two endpoints of a straight section while it is making π radian rotation within the plane so that the section always passes a fixed point P of the plane and the position of the section at 0 angle fully overlaps with its position at angle π radian; moreover to every positive distance ξ there is a positive rotational angle ε such that if $\Delta\alpha < \varepsilon$, the distance between the two points of rotation relative to the section at an arbitrary rotational angle α and at $\alpha + \Delta\alpha$, respectively, is less than ξ. Clearly, a continuous equichordal curve is a Jordan curve.

For brevity, in this paper we introduce and use the abbreviation ECC for a continuous equichordal curve.

Generally, the point P in Def. 1 is called the *equichordal point* of the curve. For a long period, it was only a conjecture – posed by Fujiwara (1916) and by Blaschke, Rothe and Weizenböck (1917) – that there is no convex equichordal curve having two equichordal points. Rychlik (1997) has proven recently that even for a wider subclass of equichordal curves, the *strongly starlike curves* (see Def. 2), it holds that there are no equichordal curves of Jordan type, having more than one equichordal point. The proof required deep and complex topological considerations. A simple corollary of Rychlik's theorem is, that if an equichordal curve has a symmetry axis, then the equichordal point must be part of it.

Definition 2. *(Strongly starlike curve)* A closed plane curve c is strongly starlike if there exists an open set of plane points S, the *set of star points*, for which it holds that $S \in C$, where C is the open set bounded by c, so that for every line l such that $l \cap S \neq \emptyset$; there exist two distinct points W and Z, such that the set $l \cap c \equiv W \cup Z$.

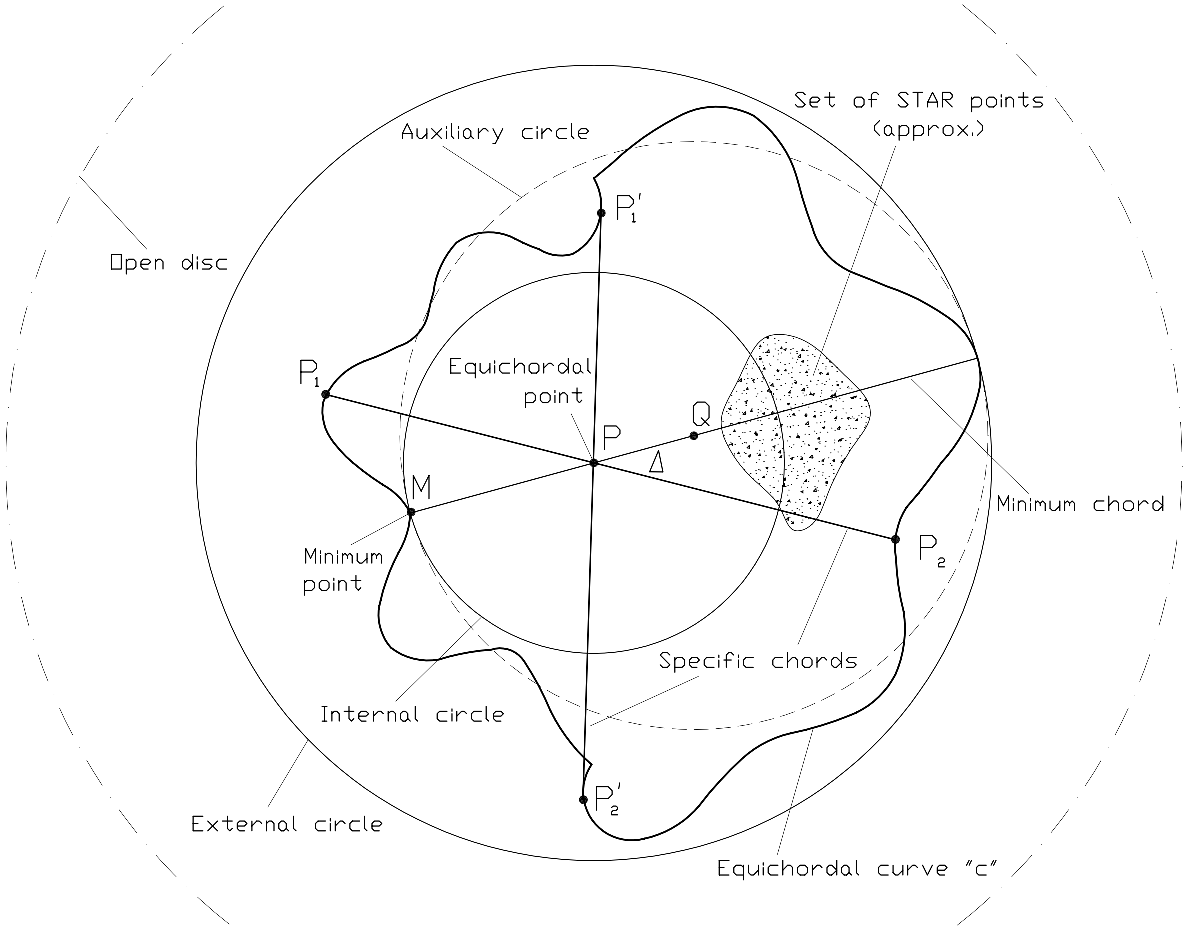

Fig. 1. A general equichordal curve and some of its features

For illustration, a general, strongly starlike ECC is depicted in Fig. 1. Note that in this case the equichordal point P is *not* a star point of the curve c (see e.g. the chord $\overline{P_1'P_2'}$, having three common points with curve c). The figure also shows some additional features which will be discussed or defined later.

By experimenting with different equichordal curves, one can observe that it is seemingly impossible to create an even-fold symmetrical equichordal curve. It is plausible that if the curve has a larger than average convex curvature at a point P_1, the curve at the other endpoint P_2 of a specific chord (see Def. 3) starting from P_1 must have a smaller than average convex (or concave, nonconvex) curvature, and also if the curve is locally nonconvex at one end of a specific chord, it must be convex at its other end.

Proposition 1. the number of symmetry axes of an equichordal curve can not be even.

Proposition 2. the number of rotational symmetries of an equichordal curve can not be even.

To illustrate the above propositions or conjectures, Fig. 2 presents a few different ECCs with multiple symmetry axes and with multiple rotational symmetries. Four of them possess D_N-type (dyadic group) symmetry (i.e. both the N-fold cyclic rotational and the N-fold mirror symmetry), except for items c) and d), which have only C_N (cyclic) symmetry.

It follows from Def. 1 that a function describing an ECC in polar coordinate system, centered at its equichordal point must obey the equation:

$$r(\varphi) + r(\varphi + \pi) = d = 2r_0 \tag{1}$$

where d is a positive constant, the length of the *specific chord* (Def. 3) of the curve.

NB *Throughout this paper, the center of the polar coordinate system and the equichordal point are chosen to be identical.*

Definition 3. *(Specific chord)* A chord of an equichordal curve containing the equichordal point is a specific chord.

It follows simply from Eq. (1) that given an arbitrary non-negative function $r_b(\varphi);\ \varphi \in [0, \pi]$ such that $r_b(0) + r_b(\pi) = d$ and $r_b(\varphi) < d;\ \varphi \in [0, \pi]$, it can be completed into the polar equation of an equichordal curve by using Eq. (1), having its equichordal point at the center of the coordinate system. The geometrical meaning of this is (see. Fig. 1) that an open plane curve c_b having the endpoints P_1 and P_2, uniquely determines an equichordal curve c, if the equichordal point P is chosen within section $\overline{P_1P_2}$, and the c_b open curve is part of the open disc, centered at P, having its radius d, equal to the length of section $\overline{P_1P_2}$ (see Fig. 1).

Note, that in accordance with the terminology of the theory of special plane curves, the equichordal curves could be called *auto-conchoids* with respect to their equichordal points, since the completing part of the equichordal curve is a conchoidal map with respect to point P and with parameter d, of the original open curve (see e.g. Lawrence (1972)). Clearly, the endpoints of any specific chord of an equichordal curve divide it into two open curve sections, being *conchoidal maps* of each other.

Definition 4. *(Internal conchoidal map)* The plane point B is the internal conchoidal map of plane point A with respect to plane point P with the

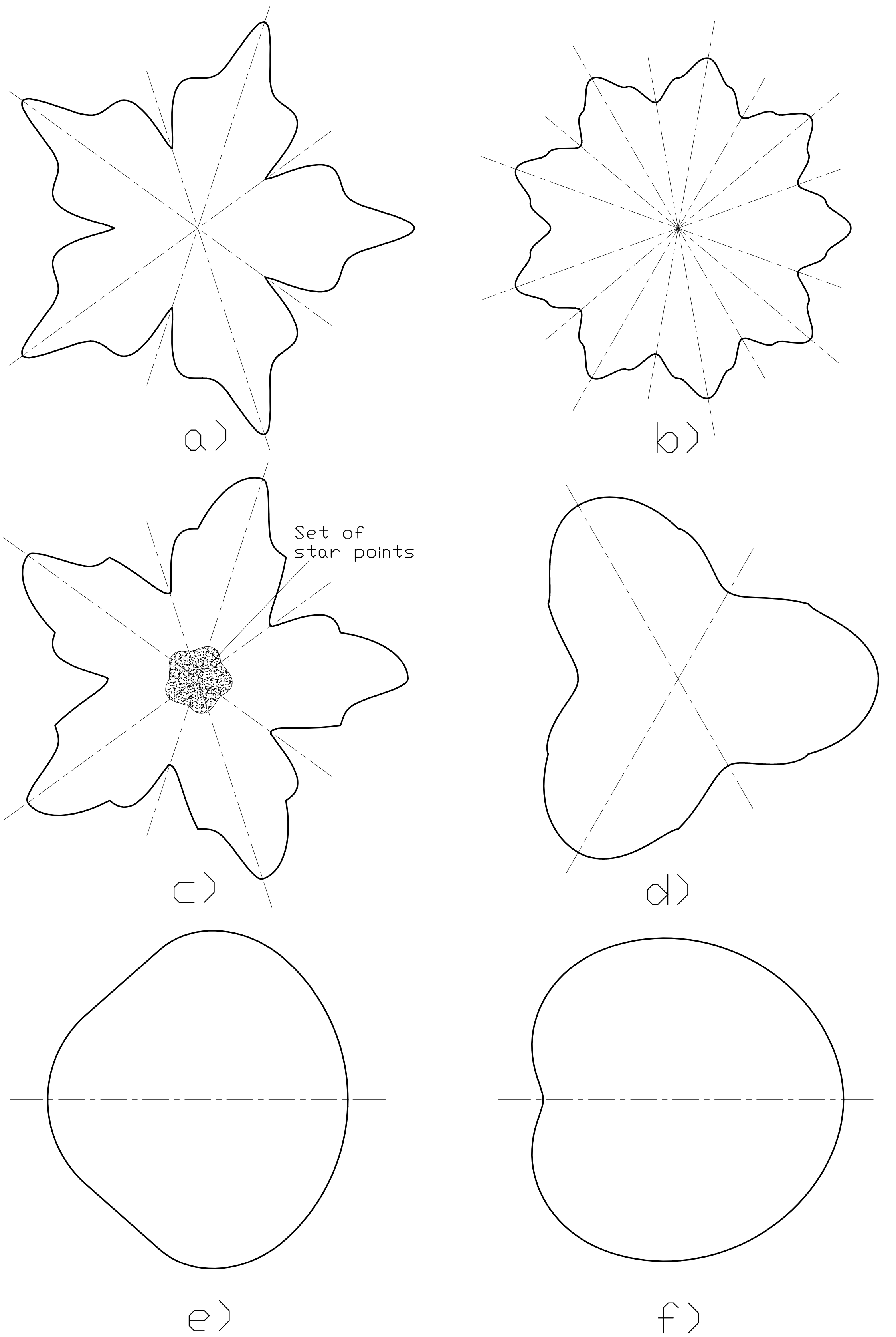

Fig. 2. A gallery of symmetrical equichordal curves

parameter d, $\mathbb{C}(A \xrightarrow{P,d} B)$, if $P \in \overline{AB}$ and $\text{dist}(A, B) = d$, where d is a positive scalar constant. Clearly, if $\mathbb{C}(A \xrightarrow{P,d} B)$ then $\mathbb{C}(B \xrightarrow{P,d} A)$.

Definition 5. *(Base section)* A curve section (a contiguous or non-contiguous subset) of an equichordal curve which uniquely determines the shape of the whole curve, is called a base section of the equichordal curve.

For a non-symmetrical curve, a base section covers a π-sized angle range, as seen from the equichordal point; for a curve having N-fold rotational (cyclic) symmetry, it covers a π/N-sized angle, while for a curve with N symmetry axes the base section covers a $\pi/(2N)$-sized angle range.

The base section of an equichordal curve together with the internal conchoidal map of the base section with respect to the equichordal point and with the parameter of the length of a specific chord, yield a symmetry element of the curve (in non-symmetrical case the symmetry element is the whole curve), then the appropriate symmetry transformations complete the curve.

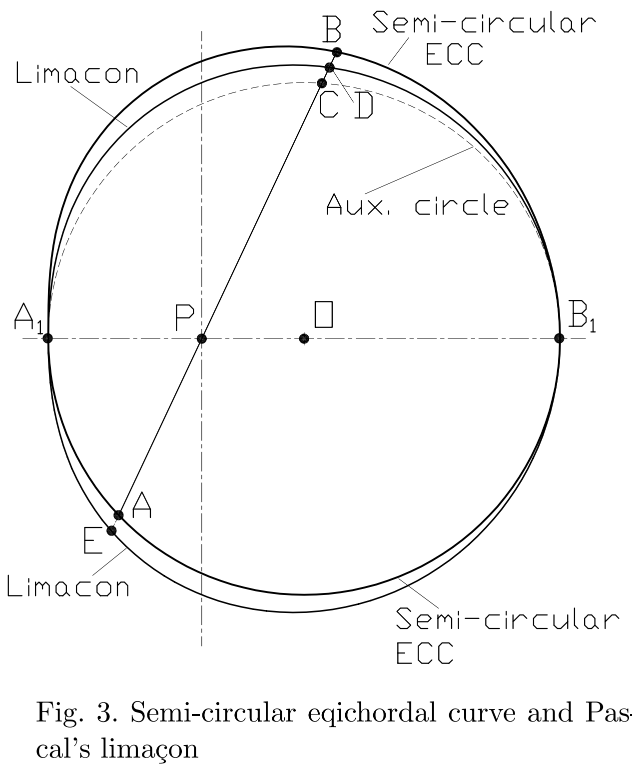

Fig. 3. Semi-circular eqichordal curve and Pascal's limaçon

Based on the considerations above, an equichordal curve can be obtained, for example, if one takes a half circle, c_b and picks a point P (other than the middle point, to avoid a trivial solution) on the diameter section determined by the two endpoints of the half circle A_1, B_1, and while one end point of the section (e.g. A_1) is pulled along the half circle so that the section always passes the selected point P; the locus of the other end point of the section makes the other half of the equichordal curve. In other words, the set of points determined by the map $\mathbb{C}(A \xrightarrow{P,d} B)$ for every point $A \in c_b$ completes the curve. This non-symmetrical equichordal curve is referred later as *semi-circular equichordal curve* (see Fig. 3). The equation of this curve, in a polar coordinate system centered at the equichordal point, can be derived by elementary methods:

$$r(\varphi) = \begin{cases} \varphi \in (0, \pi); & d + \varDelta \cos\varphi - \sqrt{(d/2)^2 - \varDelta^2 \cdot \sin^2\varphi} \\ \varphi \in [\pi, 2\pi]; & \varDelta \cos\varphi + \sqrt{(d/2)^2 - \varDelta^2 \cdot \sin^2\varphi} \end{cases} \tag{2}$$

where $\varDelta = \mathrm{dist}(P, O)$ is the distance of the equichordal point from the middle point of the closing diameter of the semi-circle. The upper equation corresponds to the non-circular part, while the lower to the circular part of the curve.

Starting from a semi-circular equichordal curve, one can create a symmetrical equichordal curve rather simply by using the next recipe (see Fig. 3):

- make the semi-circle into a full circle (which is the auxiliary circle of the semi-circular curve, according to Def. 10),
- start moving the closing diameter $\overline{A_1B_1}$ like in the previous case, but
- after every elementary movement mark the mid-point D of the section $\overline{BC}$ of the moving chord, which section falls between the completing half circle and the non-circluar section of the semi-circular equichordal curve;
- the locus of these middle points D and the corresponding points E, such that map $\mathbb{C}(D \xrightarrow{P,d} E)$, yield a symmetrical curve.

By definition $\mathrm{dist}(A, B) = \mathrm{dist}(E, D)$ and also $\mathrm{dist}(A, E) = \mathrm{dist}(B, D) = \mathrm{dist}(C, D)$, corresponding to the notation shown in Fig. 3. It can easily be derived that the resulting curve obeys the following equation:

$$r(\varphi) = d/2 + \varDelta \cdot \cos\varphi = r_0 \cdot (1 + \delta \cdot \cos\varphi) \tag{3}$$

which is nothing other than the polar equation of the rather well known Pascal's limaçon curve (see, for example, Lawrence (1972)) which is perhaps the simplest and the only *named* equichordal curve, apart from the circle.

Definition 6. *(Internal circle, minimum point)* The internal circle of an equichordal curve is the largest circle around its equichordal point, having no points outside the equichordal curve. The common points between the equichordal curve and its internal circle are called the minimum points of the equichordal curve (see Fig. 1).

Definition 7. *(External circle, maximum point)* The external circle of an equichordal curve is the smallest circle around its equichordal point, having no points inside the equichordal curve. The common points between the equichordal curve and its external circle are called the maximum points of the equichordal curve (see Fig. 1). Note that the maximum and the minimum points are *internal conchoidal maps* of each other with respect to the equichordal point.

Note that the internal and the external circles are *internal conchoidal maps* of each other relative to the equichordal point and with the parameter d.

Definition 8. *(Extremum chord)* A *specific chord* containing a *minimum point* (and also a *maximum point*) of an ECC is an extremum chord.

Definition 9. *(Eccentricity, relativ eccentricity)* The eccentricity (Δ) of an equichordal curve is the distance between the equichordal point and the middle point of an *extremum chord*. The relative eccentricity (δ) is the eccentricity divided by (r_0), the half of the specific chord length (d) (see e.g. Eq. 3).

Definition 10. *(Auxiliary circle)* A circle, drawn around the middle point of an *extremum chord* is called an auxiliary circle of the equichordal curve (see Figs. 1 and 10).

Definition 11. *(Simple equichordal curve)* An equichordal curve is called simple, if a contiguous set (curve section or a single point) contains all of its *minimum points*. A simple equichordal curve may not have more than one symmetry axis.

2.2 Convexity of the symmetrical equichordal curves

An interesting observation is that the convexity of the symmetrical equichordal curves is connected with their eccentricity: the larger the number of symmetry axes of an equichordal curve, the smaller the eccentricity can be to allow a convex curve. If we look, for example, at the series of N-fold symmetrical curves specified by the polar equations

$$r(\varphi) = r_0 \cdot [1 + \delta \cdot \cos(N\varphi)]\,; \quad N = 1, 3, 5, \ldots, 2k+1, \ldots \tag{4}$$

which can be called *N-fold limaçons* – as they are referred hereafter –, the $\delta \leq 1/(N^2+1)$ condition has to be satisfied, to have convex curves. To see this, we need the following theorem.

Theorem 1. *A closed plane curve c, which is at least two times differentiable at point A, $A \in c$, is locally convex at A if the second derivative of the polar equation of the curve is less than or equal to r_A, where r_A is the r coordinate of point A, and the center O of the polar coordinate system is on line l, which is chosen so that l is perpendicular to t and $A \in l$, where t is the tangent line of c at A, and the open section $\overline{AO} \in C$ where C is the open set enclosed by c.*

If the first derivative and both the left $(-)$ *and right* $(+)$ *side second derivatives exist at* A*, then the generalized theorem is* $\max\left\{r''_{-}(\varphi), r''_{+}(\varphi)\right\} \leq r_A$. (see Fig. 4)

Proof: By definition, a closed curve c enclosing the open set C is locally convex at a given point $A \in c$, if there exists an open disk D around A such that $C \cap D$ is convex.

It is taken as known that if a curve is differentiable two times at point A, then there exists a curvature circle c_c; $(A \in c_c)$, having the radius r_c and possessing the following features:

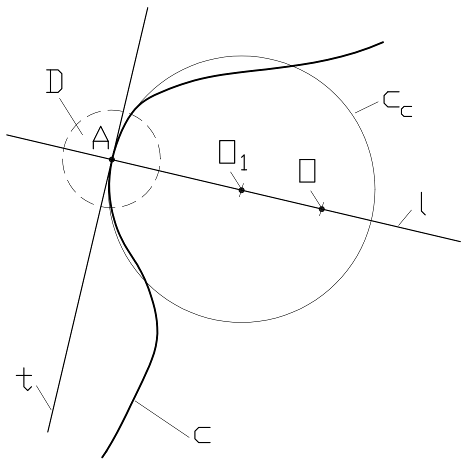

Fig. 4. Condition of local convexity

- ≫ the tangent line of c_c and of curve c at A are identical (i.e. the first derivative of the polar equations of c_c and that of c at A are identical),
- ≫ the curvature of c_c and that of curve c at A are identical: $\kappa = 1/r_c$ (i.e. the second derivative of c and c_c at A are also identical),
- ≫ c is locally convex at A if there is a $\zeta > 0$ such that if $r_{\mathrm{d}} < \zeta$ then both $C \cap D \in H_t$ and $C_c \cap D \in H_{\mathrm{t}}$, where C_c is the open set inside the curvature circle and H_{t} is one of the open half-planes supported by t, the tangent line. In other words, the curve c is locally convex at A if its curvature circle is within the *same* half plane supported by t, which contains the $C \cap D$ intersection if $r_{\mathrm{d}} < \zeta$.

The equation of line t in a polar coordinate system having its center O on line l, being perpendicular to t, is:

$$r_{\mathrm{t}}(\varphi) = r_{\mathrm{A}}\sqrt{1 + \tan(\varphi)}, \tag{5}$$

where $r_{\mathrm{A}} = \mathrm{dist}(A, O)$. Thus, its first and second derivatives are as follows:

$$\frac{\mathrm{d}r_{\mathrm{t}}}{\mathrm{d}\varphi} = r_{\mathrm{A}}\frac{\sin(\varphi)}{\cos^2(\varphi)}; \quad \frac{\mathrm{d}^2 r_{\mathrm{t}}}{\mathrm{d}\varphi^2} = r_{\mathrm{A}}\frac{1 + \sin^2(\varphi)}{\cos^3(\varphi)} \tag{6}$$

assuming that the $\varphi = 0$ direction is determined by the line $\overline{OA}$. Clearly, at $\varphi = 0$ the first derivative of the tangent line is 0, and the second derivative is r_{A}. It follows from the selection of the coordinate system that the first derivative of the equation of curve c at point A is also zero.

Let the circle c_c having its center at O_1 and its radius $r_c = \text{dist}(A, O_1)$ be the curvature circle of c at point A, that is the curvature of c at A is $\kappa = 1/r_c$. Clearly, if the center of the coordinate system is chosen at O_1, both the first and the second derivatives of the function describing the curve c will be zero at point A. Let O be such that $O \in C \cap l$, that is, another point on l within the open set C, then the transformation of a polar equation of the curvature circle c_c from the coordinate system centered at O_1 to the coordinate system centered at O can be carried out, as follows:

$$r(\varphi) = q_1 \lambda \cos(\varphi) + q_2 \sqrt{r_c^2 - \lambda^2 \sin^2(\varphi)}, \tag{7}$$

where λ is the positive distance between O and O_1, and the values of q_1 and q_2 correspond to the following table, in which $H_{t,c}$ denotes the half plane supported by t containing C_c, the open set confined by the curvature circle c_c:

	Case	q_1	q_2
1)	$O \notin (A, O_1)$ and $O \notin H_{t,c}$	$+1$	-1
2)	$O \in (A, O_1)$ and $O \in H_{t,c}$	-1	$+1$
3)	$O \notin (A, O_1)$ and $O \in H_{t,c}$	$+1$	$+1$

Since, according to the conditions of the theorem, O must be chosen so that the section $\overline{AO} \in C$, case 1) means that curve c is nonconvex at A, whereas the other two cases correspond to convexity.

Taken the second derivative of the curvature circle at A (at angle φ_A), for the three cases we get:

1) $r''(\varphi_A) = -\lambda + \lambda^2/r$ and $r_A = \lambda - r_c$, the relation $r''(\varphi_A) < r_A$ yields the $(r_c - \lambda)^2 < 0$ *contradiction.*
2) $r''(\varphi_A) = \lambda - \lambda^2/r$ and $r_A = r_c - \lambda$, the relation $r''(\varphi_A) < r_A$ yields the $(r_c - \lambda)^2 > 0$ *affirmation.*
3) $r''(\varphi_A) = -\lambda - \lambda^2/r$ being always negative, is less than the positive $r_A = r_c + \lambda$, also *affirmation.*

Clearly, if the second derivative is discontinuous at point A, the above proving procedure has to be carried out twice, by using the left and the right curvature circles respectively, and both the left $(-)$ and the right $(+)$ side derivatives should satisfy the condition. □

Theorem 1 offers a useful tool to find out the convexity limit of a curve, if one knows from some consideration which point of the curve becomes first nonconvex, when a parameter changes in one direction. Typically, the ECCs get first locally nonconvex with increasing relative eccentricity at the boundary of the set of the minimum points.

Corollary: since for an N-fold limaçon (4) the most critical points from the aspect of convexity are the minimum points, thus it follows from Theorem 1 that the convexity condition for these curves is $\delta \leq 1/(N^2+1)$.

One can call the curves having the critical values of the parameters influencing their convexity *critically convex curves*. Such critically convex N-fold limaçons are shown in Fig. 5. Note that these critically convex N-fold limaçons look like regular polygons with neatly rounded corners, but it worths to mention that this series of quasi-polygons starts with $N = 1$.

Since the N-fold limaçons probably have the simplest analitical form (4) among the equichordal curves with N-fold symmetry, one may assume that the $\delta \leq 1/(N^2+1)$ condition is an absolute limit for symmetrical curves, thus no convex N-fold symmetrical equichordal curve can have a relative eccentricity larger than $1/(N^2+1)$, i.e. 1/2 for a curve with 1-fold symmetry.

Unfortunately, this proposition does not hold generally, though some examples seem to support it. For example, curve a) in Fig. 2, having the equation for its base section

$$r(\varphi) = r_0 \left\{1 + \delta \cos\left[s \cdot \ln(N\varphi + 1)\right]\right\}; \; s = \frac{\pi}{2 \cdot \ln(\pi/2+1)} = 1.6636\ldots \quad (8)$$

will have as its convexity limit for the δ parameter: $\delta \leq \left[(N^2+1)\cdot s\right]^{-1}$. It is also interesting that for a semi-circular equichordal curve (see Eq. 2), being though non-symmetrical, has as its convexity limit for the δ parameter: $\delta \leq \sqrt{2} - 1$, which is also less than 1/2 (Note that in this case the second derivative at the minimum point is discontinuous!).

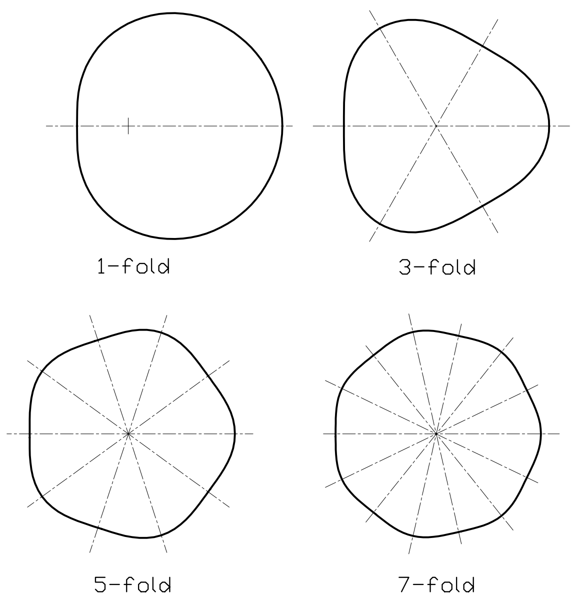

Fig. 5. Critically convex N-fold limaçons

A counterexample to the above false proposal can be created as follows. To this end, we have to introduce another subclass of equichordal curves which have an extended set of minimum points. It follows from the definition of minimum points (Def. 6), that a contiguous set of minimum points forms a circular arc having a radius identical with that of the internal circle. The set of maximum points (Def. 7), being the internal conchoidal maps (Def. 4) of the minimum points with respect to the equichordal point, also form a single or several disjoint circular arcs. The subclass of equichordal curves, having one or more extended sets of minimum points can be called *partially circular* equichordal curves. A few partially circular equichordal curves are presented in Fig. 6.

The simplest way of creating a symmetrical partially circular equichordal curve is to draw its internal circle, then to draw the specific chord perpendicular to the symmetry axis; to take then the two tangent lines from the two endpoints of this specific chord to the internal circle. If one regards the two tangent sections, together with the shorter circular arc between them as the base section of an equichordal curve, then the curve completed according to Eq. (1), it will make a *partially circular equichordal curve* with one symmetry axis (see case e) in Fig. 2, where $\delta = 0.25$). This curve will be referred to as a *tangent line partially circular* (TLPC) equichordal curve. Of course, this is not the only possibility for creating a partially circular ECC. Once we have a circular arc section with a radius of $r_0(1-\delta)$ over an angle range ω_0 and its internal conchoidal map, which is also a circular arc with the radius of $r_0(1+\delta)$ over the same angle range ω_0, we can connect the two arcs in many ways by satisfying the equichordal condition, e.g. by a sinusoid curve section.

It can rather easily be seen, that a TLPC ECC with a single symmetry axis is convex for the full range of the eccentricity parameter, $\delta \in [0, 1)$ (see e.g. case c) in Fig. 6, having a relative eccentricity $\delta = 0.95$). We had to exclude the $\delta = 1$ case since at this limit the curve becomes discontinuous, consisting of a half circle and a single, disjoint point.

Let us introduce a mapping procedure – called *N-times folding* – which is capable of creating N-fold symmetrical ECCs from 1-fold symmetrical ones.

Definition 12. *(N-times folding)* If the base section of an ECC having one symmetry axis is mapped from its original angle range $[0, \pi/2]$ into the angle range $[0, \pi/(2N)]$ so that the r coordinate of every point of the mapped base section at angle φ corresponds to the r coordinate of a point at angle $N\varphi$ in the original curve section, then the mapped base section yields an N-fold symmetrical ECC. The resulted ECC is called as the *N-times folding* of the original one.

It is noteworthy that while a TPLC ECC with one symmetry axis is always convex, the 3-times folding of the TLPC ECC is nonconvex even for small values of eccentricity (e.g. $\delta = 0.08$, see case b) in Fig. 6). It is possible to see by using elementary geometry methods that for the full range of the eccentricity parameter, $\delta \in [0, 1)$, the N-times folding of the originally convex TLPC ECC will be nonconvex.

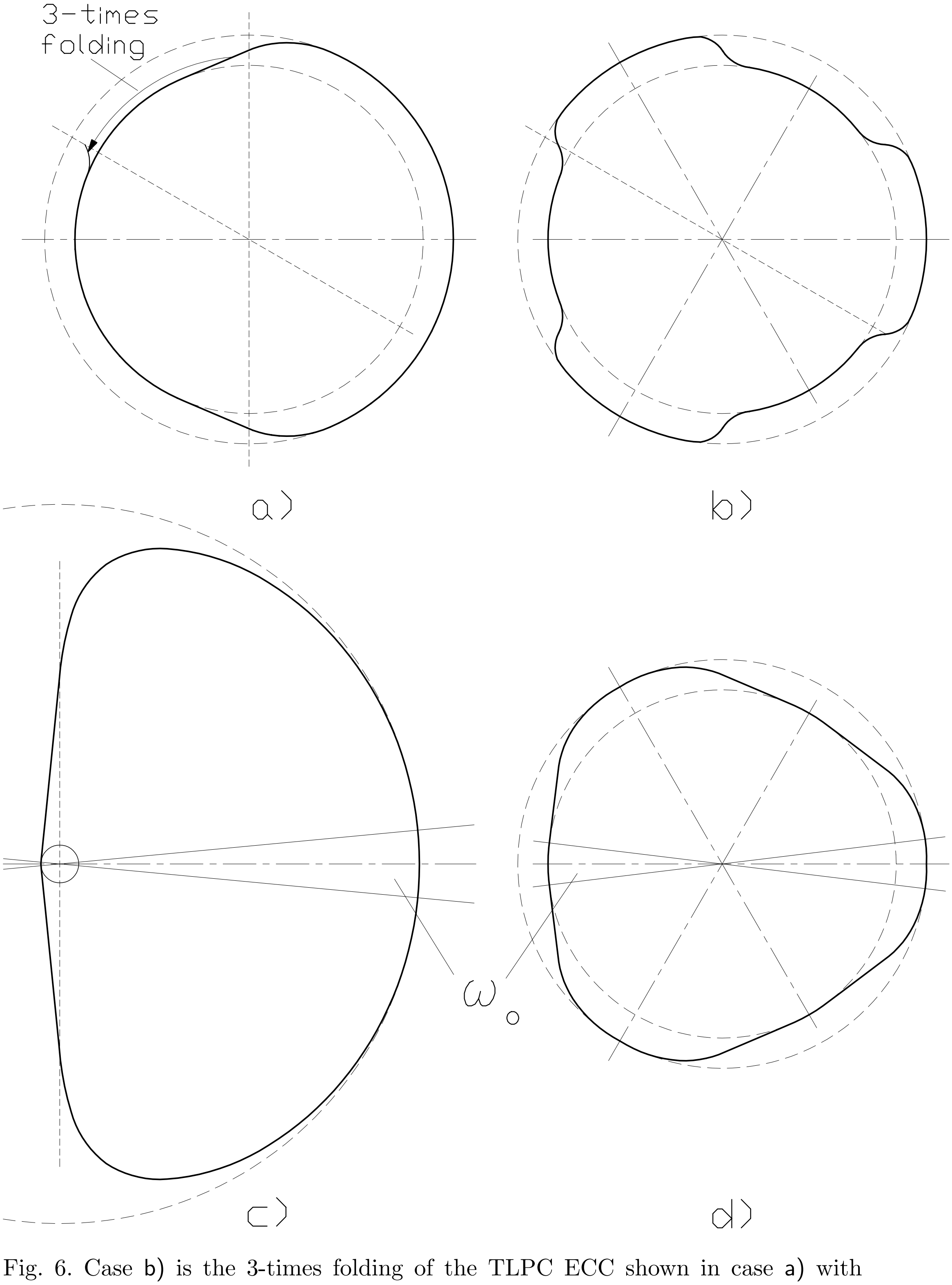

Fig. 6. Case b) is the 3-times folding of the TLPC ECC shown in case a) with a single symmetry axis, case c) is a 1-fold TLPC ECC having large eccentricity, $\delta = 0.9$, and case d) is a TLPC ECC with 3-fold symmetry. For cases a), b), and d) $\delta = 0.08$. Note that for cases c) and d), the ω_0 angles are roughly the same.

It should also be noted, that an N-times folding of a TLPC ECC is not a TLPC-type ECC itself since the map of the linear section of the TLPC curve is not linear (straight line section), i.e. the transformation *N-times folding* is not automorphous (see case a) in Fig. 6). It is, however, possible to construct TLPC ECCs with N-fold symmetry. There is a limitation, though, for the δ relative eccentricity parameter for such a curve. It can easily be seen that the base section of an N-fold TLPC ECC can not be created if $\delta > 1 - \cos[\pi/(2N)]$ since when this limit is reached, the circular arc section containing the minimum points shrinks to a single point, and the method of creating an N-fold TLPC ECC would not work for larger δ values.

Rigorously speaking, a critically convex N-fold ($N > 1$) TLPC ECC is not a partially circular ECC. The base section of such a curve can be chosen as a straight line section, extending over $\pi/(2N)$ angle range as seen from the equichordal point, and starting from the minimum point and being perpendicular to the line connecting the equichordal point and the minimum point. It is noted again that for $N = 1$, when the δ parameter reaches 1, the TLPC ECC becomes discontinuous.

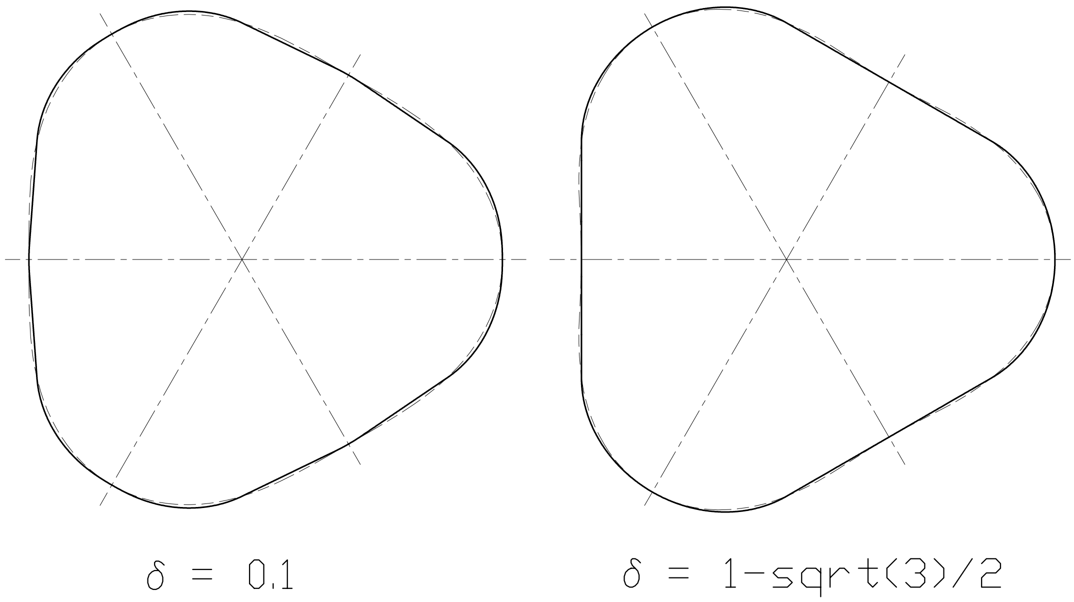

Fig. 7. Comparison of the critically convex 3-fold limaçon (dashed line) with the convex 3-fold TLPC ECC (continuous line) at $\delta = 0.1$ and the critically convex 3-fold TLPC ECC (continuous line) with the nonconvex 3-fold limaçon (dashed line) at $\delta = 1 - \sqrt{3}/2$

Figure 7 illustrates that at the δ value where the 3-fold limaçon is critically convex ($\delta = 0.1$), the TLPC ECC is still convex, whereas for the critical δ value of the TLPC ECC ($\delta = 1 - \sqrt{3}/2$) the limaçon is nonconvex.

Proposition 3. an N-fold ($N > 1$) symmetrical equichordal curve with a relative eccentricity $\delta > 1 - \cos[\pi/(2N)]$ can not be convex.

The most important facts and conjectures about equichordal curves are:

- ⇒ they can not have multiple equichordal points;
- ⇒ if they have a symmetry axis, the equichordal point is part of it;
- ⇒ they may only have an odd number of symmetry axes (unproved);
- ⇒ they may only have an odd number of rotational symmetries (unproved);
- ⇒ they can freely be constructed for a π-sized angle range around the equichordal point; or, for a $\pi/(2N)$-size angle range, for a curve with N symmetry axes;
- ⇒ an internal circle can uniquely be defined (Def. 6) to every equichordal curve as the largest circle around the equichordal point having no outer points relative to the equichordal curve;
- ⇒ the common points of an equichordal curve and its internal circle are the minimum points, an equichordal curve has at least one minimum point;
- ⇒ an external circle can uniquely be defined (Def. 7) to every equichordal curve as the smallest circle around the equichordal point having no internal points relative to the equichordal curve;
- ⇒ the common points of an equichordal curve and its external circle are the maximum points, an equichordal curve has at least one maximum point;
- ⇒ to every minimum point of an equichordal curve an auxiliary circle can be drawn containing the minimum point and being centered at the middle point of the specific chord starting at the minimum point;
- ⇒ an eccentricity parameter, Δ, can be defined for these curves, as the difference of the center of a minimum specific chord and the equichordal point; the relative eccentricity δ is defined as the ratio of the eccentricity and the half length of a specific chord;
- ⇒ an N-fold ($N > 1$) symmetrical equichordal curve can not be convex if its eccentricity exceeds some limit, which limit gets smaller as the number of symmetry axes increases; the limit is possibly equal or proportional to $\delta > 1 - \cos[\pi/(2N)]$ (unproved);
- ⇒ there is no such convexity limit for ECCs with one symmetry axis or without symmetry.

The equichordal curves may be classified in general terms, e.g. their convexity, starlikeness, symmetries, etc. From the point of view of starlikeness, it may be of special interest whether the equichordal point is a star point or not. More specific categorization can be made on the basis of the qualities of the set of minimum points (e.g. the simple ECCs, the partially circular ECCs, etc.).

3 Mechanical applications

3.1 Through vane rotary pump

The history of through vane rotary pumps dates back to the 1920s, when Parouffe (1922) filed his patent, proposing a pump, having a cylindrical cavity with a Pascal's limaçon curve cross section. As we have seen above, this curve belongs to a wider class of closed plane curves, the equichordal curves (ECC for brevity). At that time, it was almost impossible to manufacture the non-circular surface at realistic costs and with the required accuracy. Later on, a few other patents were filed, the majority of which also proposed the limaçon as the base curve for the cylindrical cavity of the pump, e.g. Seno (1991). First, in their patent specification, Ginko and Otto (1989) stated that generally any convex ECC would be adequate for such a type of pump.

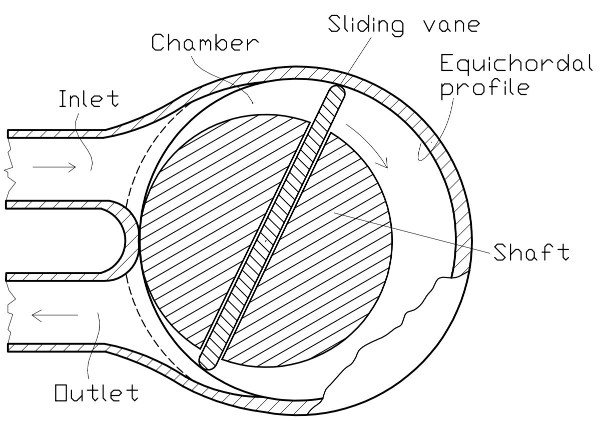

Fig. 8. Cutaway sketch of a trough vane rotary pump

The general layout of the cross section of a rotary pump with a cylindrical cavity, based on a simple equichordal curve (Def. 10), is shown in Fig. 8. The cross section of the shaft corresponds to the internal circle of the equichordal curve, thus the axis of the shaft is at the equichordal point. There is a through slot in the shaft, parallel and symmetrical to its axis. A planar vane is fitted slideably into this slot, having the length of the specific chord of the ECC. It follows from this layout, that at every angular position of the shaft, both edges of the vane are in contact (or have a constant clearance) with the cylindrical surface of the cavity. This is only true, however, if the thickness of the vane is negligible, or its end surfaces are tapered.

It is possible, however, to ensure the constant clearance between the vane ends and the cavity wall also for thick vanes with rounded end surfaces if the base curve of the cylindrical cavity is not the equichordal curve itself but its outer parallel curve, running at the distance of the rounding radius of the vane tips (it is necessary to note, that an outer parallel of an ECC is generally not an ECC). This approach is described in patents by Tanaka at al. (1983) and also by Inagaki and Sasaya (1984). Naturally, in such case, the radius of the shaft is also enlarged by the rounding radius of the vane. Figure 9 illustrates the method of *expanding* an ECC by the rounding radius of the vane tips: take an arbitrary point E of the equichordal curve, measure then in the direction of the normal vector of the curve the length of the rounding radius of the vane. This procedure has to be carried out for every point of the equichordal curve. Note that angle EPX is dependent on $\mathrm{d}r/\mathrm{d}\varphi$, if $r(\varphi)$ is the polar function defining the ECC. It is also shown in the figure that there is a maximum possible rounding radius for a vane of a given thickness τ and for every given ECC. This maximum radius is determined by the position of the vane, corresponding to the angle φ where $|\mathrm{d}r(\varphi)/\mathrm{d}\varphi|$ is the largest, because there the line in normal direction of the ECC meshes the edge of the vane at X'. It can be seen that the maximum applicable rounding radius for a vane with thickness τ, is:

$$\rho_{\max} = \frac{\tau}{2 \cdot \sin\left[\arctan\left(\left\{\left|\mathrm{d}r(\varphi)/\mathrm{d}\varphi\right|\right\}_{\max}\right)\right]}\,. \tag{9}$$

Obviously, the minimum rounding radius is $\rho_{\min} = \tau/2$.

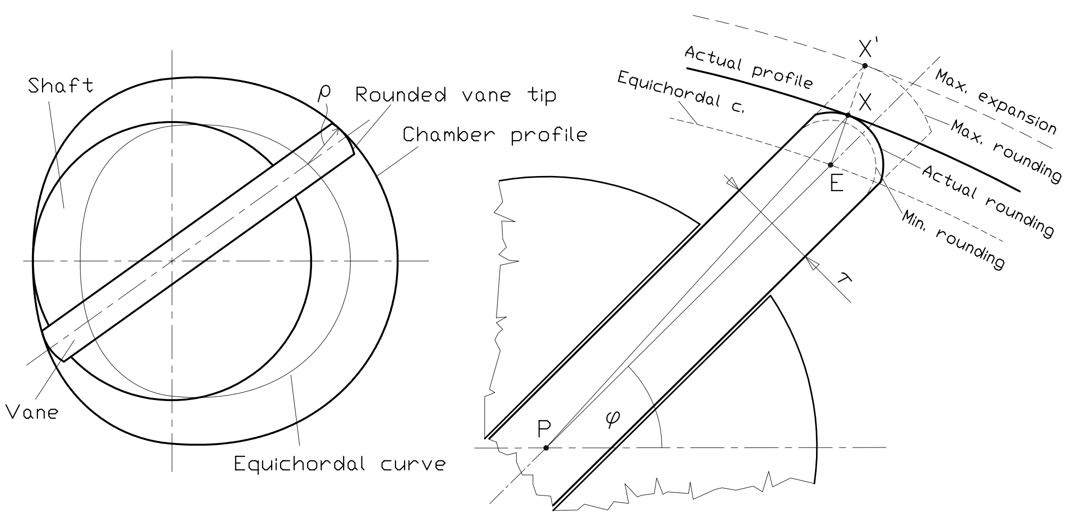

Fig. 9. Geometry of pump cavity having a realistic vane with rounded tips

The use of a large rounding radius at the vane ends offers several technical advantages (better sealing, lower friction, changing contact generatrix relative to the rounding surface of the vane, etc.), emphasizing the significance of the maximum rounding radius.

It can be seen from the general layout of a through vane rotary assembly (see Fig. 8) that the inlet and the outlet openings of the cavity are on the two sides of the contact range between the shaft and the cavity wall (corresponding to the set of minimum points of the ECC). If the working fluid is incompressible, the inlet and the outlet openings need to extend to the contact generatrices, which are determined by the position of the vane perpendicular to an extremum chord of the ECC (see points marked by N and Z in Figs.10 and 11), otherwise there would be a confined volume, which would change while the shaft rotates. Should the openings extend further, there would be a direct communication between the high and the low pressure sides thereby allowing parasitic back flow. This means that for a high precision rotary assembly working on some incompressible fluid, the inlet and the outlet openings need to extend *exactly* to the contact generatrices determined by the position of the vane, perpendicular to an extremum chord.

3.2 Constant displacement pump

It is also important to note that the pumping characteristics of such a rotary assembly, i.e. the displaced volume as a function of the angular position of the shaft, is determined by the shape of the cavity wall, and more specifically, only by the shape of the section between the two contact generatrices determined by the vane position perpendicular to an extremum chord of the ECC (see Def. 8). It is also clear that this surface section of the cavity wall corresponds to the section of the related ECC, which covers a π-sized angle range, i.e. a *base section* of the curve (see Def. 5).

This is a very important condition, since it means that the designer can create the shape of the cavity with a great degree of freedom, allowing to satisfy some specific technical requirement.

Let this requirement be the *strictly constant displacement.* At first glance, it seems impossible to satisfy this, since it is clear from Fig. 8, that when the vane is close to its vertical position, the displacement is about half as much as the displacement produced around the horizontal position of the vane. The solution is only possible if we use two cavities, which work in parallel, having a $\pi/2$ phase shift between the angular positions of the vanes in the cavities. If one thinks of the relation $\sin^2\vartheta + \cos^2\vartheta \equiv 1$, it looks obvious to chose the base section of the controlling curve of the cavities as

$$r(\varphi) = r_0 \cdot \left(1 + \delta \cdot \cos^2\varphi\right) \tag{10}$$

which yields a fairly good solution since, due to the $\pi/2$ phase shift, the parallel cavity can be regarded as if it were $\tilde{r}(\varphi) = r_0 \cdot \left(1 + \delta \cdot \sin^2\varphi\right)$ shaped. In the followings we call this curve a *cosine-square ECC.* As it is shown below, this is still not a precise solution.

For an arbitrary ECC, the change of the area A – confined by the ECC, its internal circle, a specific chord $\overline{P_1P_2}$, and the chord $\overline{NZ}$ perpendicular to the extremum chord of the ECC (see Fig. 10, top) – per an elementary change of the angle φ of the chord is (i.e. the displacement of a single cavity per

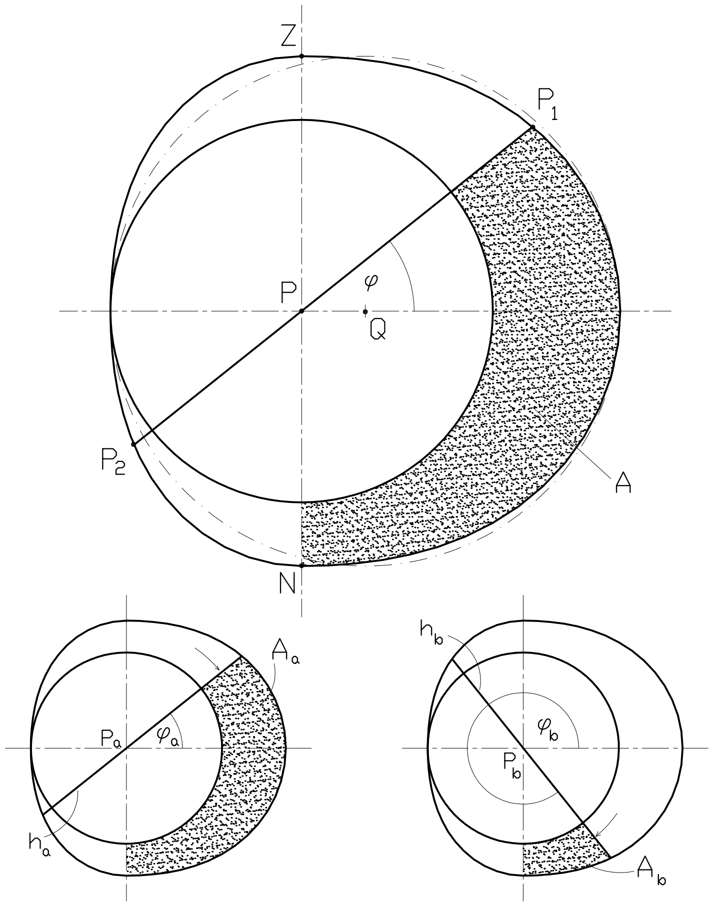

Fig. 10. The TCD curve and its basic feature

elementary rotation of the shaft is):

$$\mathrm{d}A = \frac{r^2(\varphi) - r_0^2(1-\delta)^2}{2} \cdot \mathrm{d}\varphi \tag{11}$$

That is, for the cosine-square ECC it takes the form of:

$$\mathrm{d}A = \frac{r_0^2\left[1 + 2\delta\cos^2\varphi + \delta^2\cos^4\varphi - (1-\delta)^2\right]}{2} \cdot \mathrm{d}\varphi\ . \tag{12}$$

and if we sum up the displacement of the two chambers with a phase shift of $\pi/2$, we get:

$$\mathrm{d}A_{\mathrm{sum}} = \frac{r_0^2\left[6\delta + \delta^2\left(\cos^4\varphi + \sin^4\varphi - 2\right)\right]}{2} \cdot \mathrm{d}\varphi \tag{13}$$

thus, the displacement of such a twin chamber arrangement is not constant.

If, however, we choose the base section of our ECC as

$$r(\varphi) = r_0 \cdot \sqrt{1 + (2\delta + \delta^2)\cdot\cos^2\varphi}, \tag{14}$$

the sum of the displacement of the two chambers will be:

$$\mathrm{d}A_{\mathrm{sum}} = \frac{r_0^2\,(6\delta + \delta^2)}{2} \cdot \mathrm{d}\varphi = \tilde{c} \tag{15}$$

being truly constant. From this peculiarity, we call the ECC with the base section defined by (14), a TCD curve, from *Twin Constant Displacement*. Since $\mathrm{d}A_{\mathrm{sum}}/\mathrm{d}\varphi$ for the TCD curve is constant, strict proportionality of the displacement holds with the rotational angle of the shaft. Figure 10 (top) illustrates the TCD curve along with its auxiliary circle to emphasize its shape; the two smaller figures (bottom) show two such curves around the equichordal points P_{a} and P_{b} with the chords h_{a} and h_{b}, respectively, being perpendicular to each other. The sum of the hatched areas A_{a} and A_{b} changes at a constant rate while the two perpendicular chords (h_{a} and h_{b}) turn simultaneously at a constant speed.

For a realistic pump (or rotary assembly), the TCD curve (14) is still not perfect for our goal since – in reality – such an assembly can not have infinitely thin vanes. The finite thickness of the vanes destroys the constant displacement in two ways:

- the ECC – as we have seen – has to be expanded by the rounding radius of the vane, and the expanded curve will not have the same qualities as its ECC parent;
- as the vane moves back and forth in the slot of the shaft, it works like a piston, producing some additional displacement.

The required strictly constant (pulsation free) displacement under realistic conditions can still be satisfied, if one can find the shape of the necessary ECC. Figure 11 shows a sketch of the cross section of a realistic twin chamber pump. The two chambers have a common shaft and a wall parallel to the plane of the drawing separates them (it is drawn as if it were transparent). Vane V1 moves in the upper chamber while the perpendicular vane V2 moves in the lower chamber. The vanes are displayed in two different angular positions of the shaft at an angle $\Delta\varphi$ apart from each other (the second position is drawn by dashed lines). Note that the contact point (generatrix) between the vane tip and the chamber wall (based on the parallel curve of the ECC, labeled by EECC in the figure) changes relative to the vane as the shaft rotates. The points Z and N mark the contact points corresponding to the upright position of a vane, which are also the confinement boundaries for the inlet and outlet openings, respectively.

The task is to find a curve (to be used as the base section of a symmetric and simple ECC) for which it is true, that the displacement in the two chambers – of identical cylindrical cavities with the cross section of the outer parallel of that ECC at the distance of the rounding radius of the vane tips – is proportional to the rotational angle of the shaft. To this end, it is also necessary to take into account the piston effect caused by the sliding movement of the two vanes with finite thickness. This requirement is equivalent to $A_1 + A_2 - (A_3 - A_4) = \hat{c} \cdot \varphi$, where $\hat{c}$ is a constant, the other notations correspond to Fig. 11.

Obviously the conditions described above and illustrated in Fig. 11 can be represented by a differential equation. Let us name the curve, providing the true solution of the problem, as RTCD from *Real Twin Constant Displacement*. It can be seen, that the following delayed type first order differential equation takes care of the majority of the effects, (it only neglects the change of the narrow and small volumes between the rounded tips of the vanes and the cavity wall):

$$\left[\frac{\mathrm{d}r(\varphi)}{\mathrm{d}\varphi} + \frac{\mathrm{d}r(\pi/2-\varphi)}{\mathrm{d}\varphi}\right] \cdot \tau + \\ + \left[r^2(\varphi) + r^2(\pi/2-\varphi) + 2\rho\Big(r(\varphi) + r(\pi/2-\varphi) + \rho\Big)\right] = \hat{c} \tag{16}$$

where τ is the width of the vanes, and ρ is the rounding radius of the vane tips. The first term takes care of the displacement caused by the piston-like movement of the vanes; the second term handles the displacement caused by the turning movement of the two vanes. The symmetry of the curve is utilized by substituting the argument, corresponding to the second chamber vane position, $(\varphi - \pi/2)$ by $(\pi/2 - \varphi)$ and also by solving the equation only for the $[0, \pi/2]$ range, with the boundary conditions $r(0) = r_0\,(1+\delta)$ and $r(\pi/2) = r_0$ (the $\varphi = 0$ angle means now the horizontal right direction). For obvious technical reasons we need a convex and smooth curve (having continuous first derivative everywhere). Thus, due to symmetry reasons, we need to have zero derivative at $\varphi = 0$, i.e. $r'(0) = 0$. The convexity requirement according

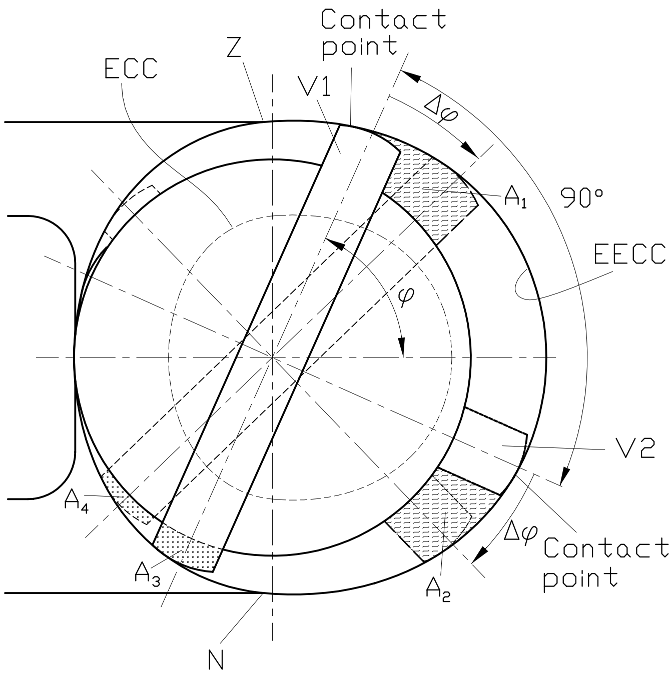

Fig. 11. Requirements for an RTCD curve taking into account the effects of the finite width of the vane

to Theorem 1 means that $r''(0) < r_0 \cdot (1+\delta)$, which has to be satisfied together with the zero first derivative. Since the parameter $\hat{c}$ is still undetermined, these conditions can be satisfied by tuning $\hat{c}$.

Our curve, apart from the r_0 scaling parameter, has three parameters: $\vartheta = \tau/r_0$, $\sigma = \rho/r_0$, and $\delta = \Delta/r_0$. Since the RTCD curve also depends on these parameters, we call them RTCD parameters. Along with the conditions of smoothness and convexity, these parameters determine the solution of (16) uniquely.

We have seen that the range of σ is fully determined by ϑ and δ by (9), but these latter two can be determined independently. It seems a rather cumbersome task to find out the range of the RTCD parameters, in which the problem might have a solution. According to the experiences, however, there exists a good range of these parameters, where the solution exists and which well suits the practical applications.

The analytitic solution of (16) seems problematic, so we have chosen to solve it by a numerical method. We discretized the equation in the $[0, \pi/2]$ range by taking about 1000 angular meshes. The numerical method applied uses two embedded iterations: the internal iteration solves the equation with a fixed $\hat{c}$ parameter by a standard relaxation method, while in an external iteration the parameter $\hat{c}$ is tuned until the first derivative at $\varphi = 0$ becomes zero and the local convexity condition is also satisfied. Fortunately, by using a central finite difference scheme, the relaxation matrix of the resulting linear system will be tri-diagonal, thus the solution is very fast and stable. As an initial guess for the function, we use the corresponding TCD curve as well as $\hat{c} = \tilde{c}$ as defined by (15). The resulted curve can be approximated with a high accuracy by a small number of circular arc sections or by linear spiral sections, both are preferable for CAD/CAM processes.

By taking into account all the details shown in Fig. 11, one can create a *control program* to calculate the displacement produced by a twin chamber rotary assembly – having cavities of any given shape and using vanes of a given geometry – as a function of the rotational angle of the vane. By using such a program, we could see, for example, that the solution of (16) with parameters of $\delta = 0.2$, $\vartheta = 0.3$ and $\sigma = 0.6$, the amplitude of the residual pulsation is about 2%, thus it provides a fairly good approximation for the RTCD problem. Should this solution be unsatisfactory for some application, it can be refined further by adding an appropriate adjustment function, e.g.:

$$r_{\mathrm{adj}}(\varphi) = \alpha \cdot \sin^{\eta}(2\varphi)\,, \tag{17}$$

to the function solving (16) and fit the two parameters α and η so that to obtain the smallest pulsation. By this method, we could achieve a residual pulsation amplitude of less than 0.02% for the above RTCD parameter values, thus obtaining an almost perfect solution of the RTCD problem. Similarly effective correction could be achieved for a wide range of the RTCD parameters by using the function (17) with appropriately fitted parameters. As an alternative, one could set up and solve a differential equation taking into account all the details shown in Fig. 11.

3.3 Other possible applications

A rotary assembly similar to the one described in the previous section could also be applied for purposes other than a pump, e.g. hydraulic/pneumatic motor. Furthermore, mechanical componentss having a cross section of some equichordal or a related curve (e.g. a parallel) may be useful in many different constructions, e.g. in special transmissions. From among the numerous possible applications, we only give a superficial description of a setup, applicable as a rotary internal combustion engine.

This application is interesting mainly because here the designer has also an opportunity to utilize effectively the freedom of constructing the shape of the base section of an equichordal curve. Here a rotary assembly, similar to that shown in Fig. 8 (but having a thicker vane with a relatively large rounding radius) is applied as the combustion chamber of an internal combustion engine. Figure 12 depicts a possible setup of a rotary internal combustion engine in three cross sections, applying two rotary assemblies (top and bottom pictures), working on the same shaft,

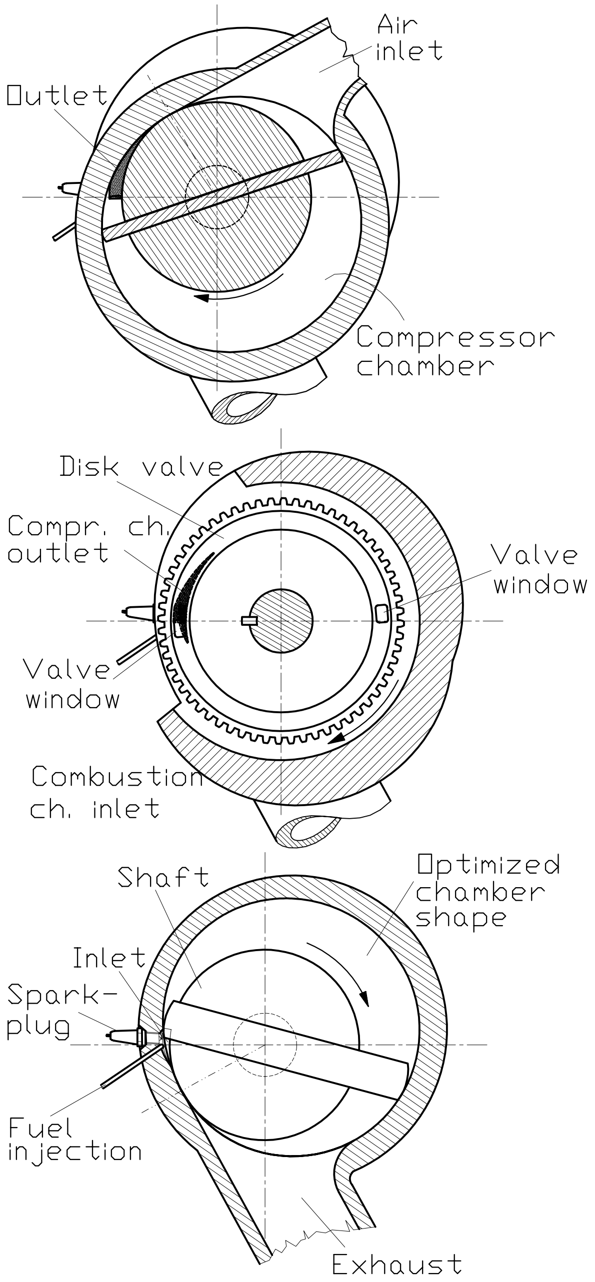

Fig. 12. Layout of a rotary internal combustion engine in three cross sections

one acting as the compressor, the other as the combustion chamber. The two assemblies rotate at the same speed with a fixed phase shift between the vane positions. The phase shift is set so that when the vane in the compressor chamber approaches the minimum point, the vane in the combustion chamber has just passed the minimum point, thus the first chamber holds compressed air, while the other starts to create vacuum. At this moment a valve, e.g. a rotary disc valve (see middle figure of Fig. 12) connects the outlet opening of the compressor with the inlet opening of the combustion chamber, so the compressed air from the first, shrinking chamber is pressed into the expanding combustion chamber. While the air is blowing through the openings, a fuel injection nozzle injects the necessary amount of fuel into the combustion chamber. When the air transfer has been completed, the valve closes and a spark plug ignites the air-fuel mixture. The expanding burning mixture exerts torque on the shaft by means of the sliding vane of the combustion chamber.

It is at this point, that the shape of the chamber plays an important role. While the time-profile of the expansion rate in the combustion chamber of traditional piston engines practically can not be modified, it can easily be tuned in this set up. To achieve optimum burning conditions, thus good fuel performance, it would be very desirable to adjust the expansion time profile according to the characteristics of a given fuel type. Based on the considerations explained in the previous sections, such an accurate tuning is certainly possible for such a rotary engine having ECC based chambers, if someone knows the requirements for the optimal expansion profile for a given fuel type.

The equichordal curves with multiple symmetries may also have some practical applications in mechanics. In such applications, apart from utilizing the equichordal feature of the curves, the fine details of their shape may also have significance, which can be tuned to satisfy some required condition.

4 Summary

The class of general equichordal curves can be introduced as a single step generalization of the circle. After the most enigmatic conjecture concerning the equichordal curves are proven, a few interesting further conjectures are proposed in this context. A generalization of Pascal's limaçon curve, the N-fold limaçons, and the notion of critical convexity have been introduced. Some hints are proposed for classifying the different equichordal curves. The freedom in constructing the base section of an equichordal curve can be utilized in a number of precise and promising mechanical applications, using some well tuned shape of the curve to satisfy some special requirement at a high accuracy. One of these possible applications is a highly accurate pulsation free positive displacement pump. For practical applications the curves are smooth so they can be well approximated either by consecutive circular arc sections, or by linear spiral arc sections, which ensure good manufacturability by CAM devices.

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