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## THE NUMBER OF PRIMES WITH A GIVEN NUMBER OF BINARY DIGITS

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ABSTRACT. Numerical result gave hint that the number of primes with  $n$  binary digits forms a monotonically increasing sequence. By using Dusart's upper and lower bounds for the prime counting  $\pi(k)$  arithmetic function, it is shown that  $\pi(4k) - \pi(2k) \geq \pi(2k) - \pi(k)$  for every  $k$ , positive integer. The monotonically increasing sequence of the number of primes with a given number of binary digits is a special case of this assertion.

### 1. INTRODUCTION

The number of primes with 1, 2, 3, ... binary digits can be determined by a fairly simple computer program for a not too high number of digits. The first 22 elements of this sequence is as follows:

$$(1.1) \quad 0, 2, 2, 2, 5, 7, 13, 23, 43, 75, 137, 255, 464, 872, 1612, 3030, 5709, 10749, 20390, 38635, 73586, 140336, \dots$$

Apart from its first two elements, it is identical to the sequence defined as the number of primes in  $[2^n, 2^{n+1}]$ , as defined by E. Labos at [1].

### 2. NUMBER OF PRIMES IN $[k, 2k]$

The sequence in (1.1) obviously suggests a proposition – which is substantially sharper than the well known Bertrand's postulate – that the number of primes in  $[k, 2k]$  is less than or equal to the number of primes in  $[2k, 4k]$  for an arbitrary positive  $k$ . We will see that by using Dusart's [2] results for upper and lower bounds for the prime counting  $\pi(k)$  function, the proposition can be proved. Dusart claims in [2] that for  $x \geq 599$

$$(2.1) \quad \frac{x}{\log x} \left( 1 + \frac{0.992}{\log x} \right) \leq \pi(x) \leq \frac{x}{\log x} \left( 1 + \frac{1.2762}{\log x} \right)$$

where the upper bound holds for all positive values exceeding 1.

**Proposition 2.1.** *For every  $k > K$ ,  $\pi(4k) - \pi(2k) \geq \pi(2k) - \pi(k)$ , where  $K$  is a sufficiently large positive integer.*

*Proof.* Let denote the Dusart's lower bound for  $\pi(x)$  in Eq. (2.1) by  $\ell(x)$  and the upper bound by  $u(x)$ . Hence, it is enough to show that  $f(x) = \ell(4x) + \ell(x) - 2u(2x) \geq 0$  for every  $x > X > 0$ , where  $X$  is some positive integer. By introducing  $z = \log x$  as a new variable, we can re-write

$$(2.2) \quad f(x) = \varphi(z) = 4e^z \left[ \frac{z + 2 \log 2 + \delta}{(2 \log 2 + z)^2} + \frac{z + \delta}{4z^2} - \frac{z + \log 2 + \varepsilon}{(\log 2 + z)^2} \right]$$

where we introduced  $\delta = 0.992$  and  $\varepsilon = 1.2762$  for the two constants, proposed by Dusart. Clearly, all the three rational function terms within the bracket converge to 0 as  $z$  approaches infinity.

If we can see that the derivative of the three terms together is negative for  $z > Z$  for some fixed  $Z$ , then the claim is proved.

To this end we introduce  $\vartheta(z)$  so that  $\varphi(z) = 4e^z \vartheta(z)$ , and re-write  $\vartheta(z)$  into a single proper fractional rational function. Instead of writing up the polynomials in the nominator and in the denominator, it is enough to determine their degree and the coefficients of the highest degree terms. The nominator will be of degree 5 and three such terms are produced but two of them cancel each other, so there remains only a single term with coefficient 1. Clearly, the coefficient of the sixth degree term in the common denominator will be 4.

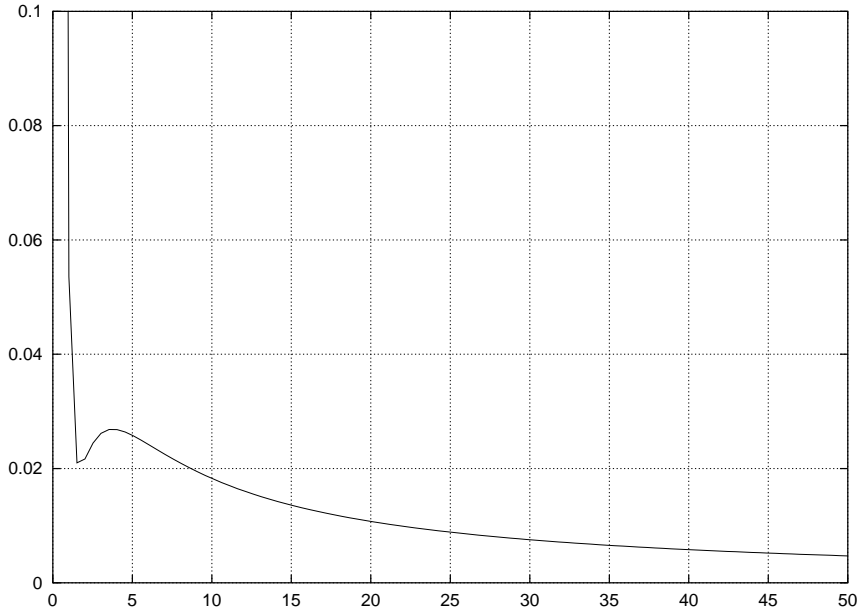


FIGURE 2.1. The function  $\vartheta(z)$  for positive values

Taking the derivative of  $\vartheta(z)$ , the denominator of  $\vartheta'(z)$  will be of degree 12. In the the nominator of  $\vartheta'(z)$  there will be two terms of degree 10, one from the derivative of the nominator as multiplied by the denominator:  $5z^4 \cdot 4z^6$  and the other from the derivative of the denominator multiplied by the nominator with negative sign:  $-24z^5 \cdot z^5$  thus the coefficient of the degree 10 term is  $-4$ . This means that there exists a  $Z > 0$ , such that if  $z > Z$  then  $\vartheta'(z) < 0$ . As  $\vartheta(z)$  converges to 0 for large  $z$ , this means that if  $z > Z$  then  $\vartheta(z) > 0$ .  $\square$

*Remark 2.2.* The function  $\vartheta(z)$  is plotted in Fig. 2.1, and it indicates that the highest root of  $\vartheta'(z)$  is around 4. By calculating the polynomial in the nominator of  $\vartheta'(z)$  in detail, it can be seen that in fact it is the largest root. (The largest coefficient of the polynomial is 35.986568 – the coefficient of the 2-nd order term –, thus no root exists with an absolute value beyond 37. But at 37, the polynomial is positive, and we know that it converges to 0.) Thus  $Z = 4$ , i.e.  $x = 55$  is a proper value. This means that for  $x > 55$ , a  $>$  sign can be used in the proposition instead of the  $\geq$  sign. Also note that Prop. 2.1 asserts more than the sequence given in the Introduction is monotonic, since it also tells that *the gaps of every geometric series with quotient 2* contain monotonically increasing number of primes, at least except a finite number of cases. Also note that the values of  $\delta$  and  $\varepsilon$  were not used in the proof except for the determination of the highest root of  $\vartheta'(z)$ .

*Remark 2.3.* Due to the limitations of Dusart's lower bound for estimating  $\pi(x)$ , we restricted ourselves to larger values. It can be seen though very easily, that the assertion in Proposition 2.1 holds also for values below 599. For example by the PARI/GP [3] program below (or a similar one) one can verify it:

```

/* # of primes in the gaps of geometric progressions with q=2 */
for(j=3,599, print("*** j"); j0=j;n0=0; forprime(i=j0,2*j0,n0+=1);
    print(2*j0" n0); j0*=2;
    while(j0<1200, n1=0; forprime(i=j0,2*j0,n1+=1);
        if(n1<n0,print(">>>> j" j0" n0" n1));
        print(2*j0" n1;n0=n1; j0*=2) } .

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Note that for base 2 (1.1) covers the range below 600.

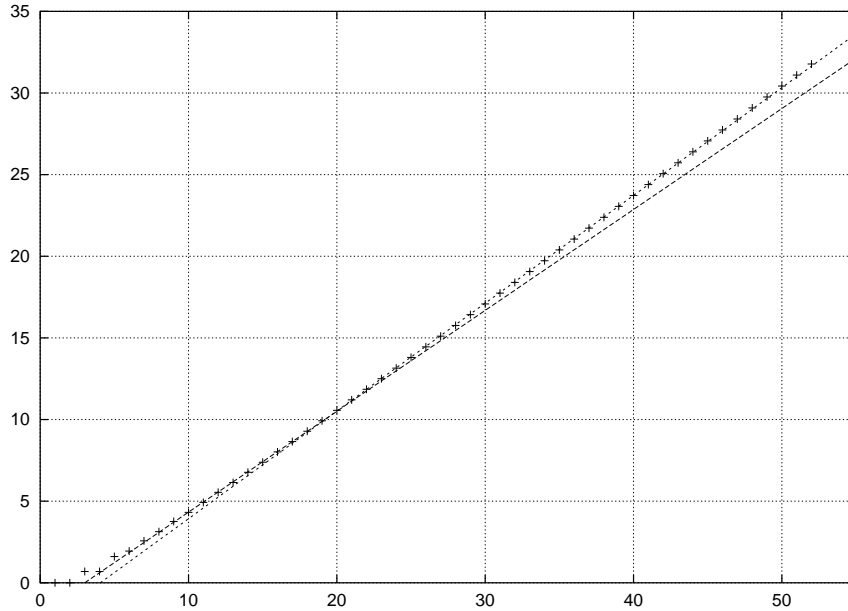


FIGURE 2.2. The number of primes of given number of binary digits and the function  $e^{c(x-3)}$  in logarithmic scale

**Theorem 2.4.** For every  $k > 0$  integer,  $\pi(4k) - \pi(2k) \geq \pi(2k) - \pi(k)$ .

*Proof.* See the proof of Proposition 2.1 and the remarks above. □

### 3. CLOSING REMARK

It is a remarkable observation, that if one plots the logarithm of the number of primes in  $[2^{x-1}, 2^x]$  as function of  $x$  positive integer values (the sequence in (1.1)), then the points seem to be close to the line  $y = c \cdot (x - 3)$ , where  $c \simeq \frac{\sqrt{5}-1}{2}$ , i.e. the “golden ratio” (the continuous line in Fig. 2.2). However, the terms of the sequence beyond 22 are systematically above that line. Terms up to the  $52^{\text{nd}}$  could be obtained from [4], which presents the values of  $\pi(2^n)$ . If one fits straight line by linear regression to the points, it can be seen that the coefficient of the first order term is systematically increasing, when points are omitted one-by-one from the left side. The steeper line has been fitted from the 15-th point on.

Re-visiting the Dusart’s lower and upper bound functions for  $\pi(x)$  (see Eq. 2.1), in Fig. 2.3 one can see how well the  $u(x)$  and  $\ell(x)$  functions (as they were defined in the proof of Prop. 2.1) approximate the prime counting function. The last positive value of the  $(\ell(x) - \pi(x))/x$  function is at 598. It is also obvious that Dusart has meant  $\lceil \ell(x) \rceil \leq \pi(x) \leq \lfloor u(x) \rfloor$  when he used the  $\leq$  sign. By examining further the same curves (up to 20000), it seems obvious that the constant  $\delta = 0.992$  in the  $\ell(x)$  function is rather strongly under estimated. If one uses a value as high as  $\delta = 1.12$ , still the  $(\ell(x) - \pi(x))/x$  remains negative in the range of [3456, 20000] (see Fig. 2.4). This suggests that by using the same technique as Dusart did, better constants could be determined for the same formulae.

### REFERENCES

- [1] E. Labos: A036378, in [4].
- [2] P. Dusart: The  $k^{\text{th}}$  prime is greater than  $k(\ln k + \ln \ln k - 1)$  for  $k \geq 2$ , Mathematics of Computation, Volume 68, Number 235, January 1999, Pages 411-415

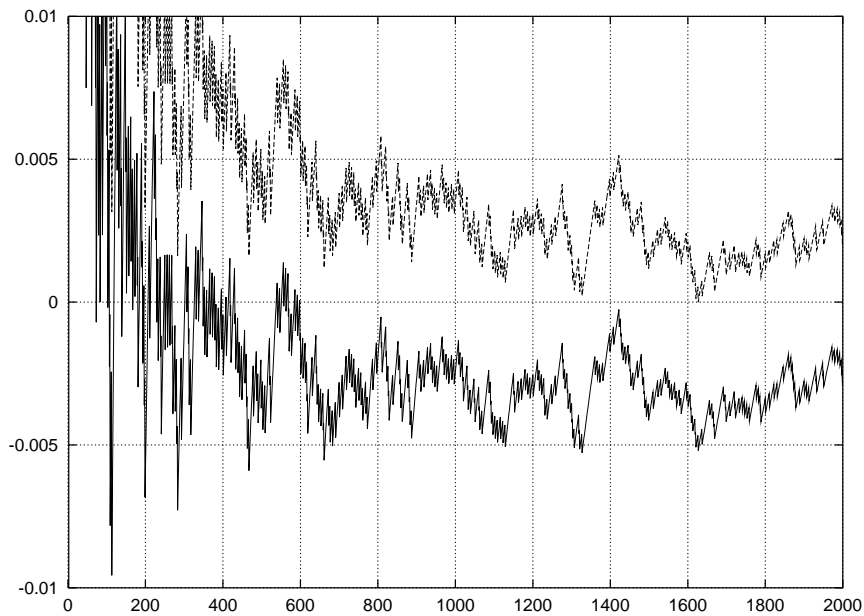


FIGURE 2.3. Visualization of Dusart's upper and lower bounding functions for  $\pi(x)$  such a way that the upper curve is  $[u(x) - \pi(x)]/x$  and the lower is  $[\ell(x) - \pi(x)]/x$ .

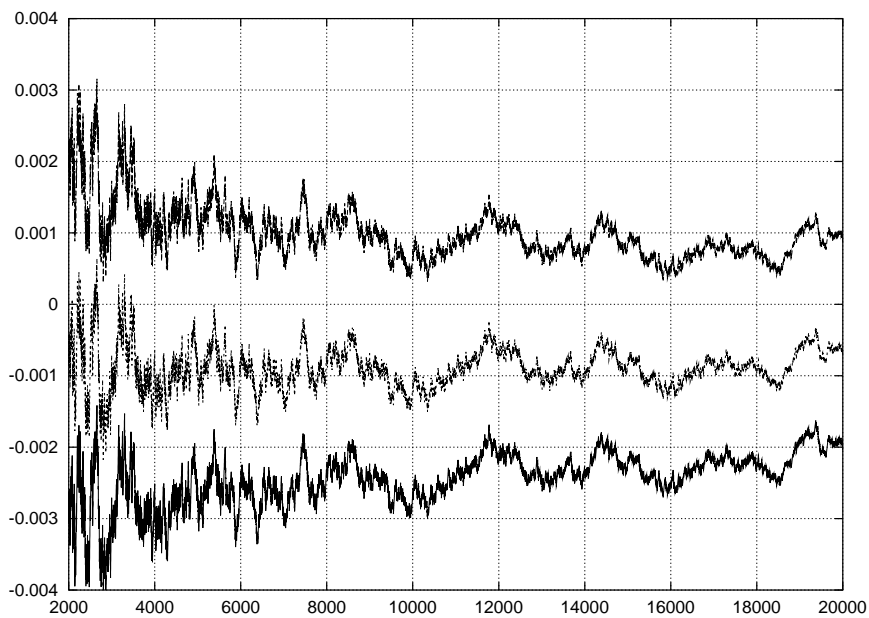


FIGURE 2.4. The continuation of Fig. 2.3, with the middle curve added, which is also  $(\ell(x) - \pi(x))/x$ , but with  $\delta = 1.12$ .

- [3] C. Batut, K. Belabas, D. Bernardi, H. Cohen, M. Olivier: User's guide to PARI/GP, <ftp://megrez.math.u-bordeaux.fr/pub/pari/>
- [4] N. J. A. Sloane, editor, 2004: *The On-Line Encyclopedia of Integer Sequences*, <http://www.research.att.com/~njas/sequences/index.html>